A Sketch of Nash's Theorem from Fixed Point Theorems

Joseph Chuang-Chieh Lin

Dept. CSIE, Tamkang University, Taiwan

Reference

- Lecture Notes in 6.853 Topics in Algorithmic Game Theory [link].
- *Fixed Point Theorems and Applications to Game Theory.* Allen Yuan. The University of Chicago Mathematics REU 2017. [link].
  - REU = Research Experience for Undergraduate students.

Outline

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The Setting

- A set N of n players.
- Strategy set $S_i = \{s_i(1), \ldots, s_i(k_i)\}$ for each player $i \in N$, k_i is bounded.
- Utility function: u_i for each player i.
- $\mathbf{\Delta}$: a Cartesian product of simplices $(\Delta_i)_{i \in N}$.
  - For $x \in \mathbf{\Delta}$, $x_i(s)$ denotes the probability mass on strategy $s \in S_i$.
  - Say $\Delta_i = \{\lambda_1 s_i(1) + \lambda_2 s_i(2) \ldots + \lambda_{k_i} s_i(k_i) \mid \lambda_i \geq 0 \; \forall i; \sum_i^{k_i} \lambda_i = 1\}$.

Nash's Theorem

Nash (1950)

Every game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ has a Nash equilibrium.

- **Note:** $u_i(\boldsymbol{x}) := \sum_{s \in S_i} x_i(s) \cdot u_i(s; \boldsymbol{x}_{-i})$.

Brouwer's Fixed Point Theorem

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Let D be a convex, compact (closed and bounded) subset of the Euclidean space. If $f : D \mapsto D$ is continuous, then there exists $x \in D$ such that

$$f(x) = x.$$

- **Idea:** We want the function f to satisfy the conditions of Brourwer's fixed point theorem.

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- Try to relate utilities of players to a function f like above.

The Gain function

Gain

Suppose that $\boldsymbol{x}' \in \boldsymbol{\Delta}$ is given. For a player i and strategy $s_i \in S_i$, we define the gain as

$$\mathrm{Gain}_{i,s_i}(\boldsymbol{x}') = \max\{u_i(s_i; \boldsymbol{x}'_{-i}) - u_i(\boldsymbol{x}), 0\},$$

which is non-negative.

- $\boldsymbol{x}'_{-i} := (x_j)_{j \in N}$, $(\boldsymbol{x}_{-i}, x_i) = \boldsymbol{x}$.
- It's equal to the increase in payoff for player i if he/she were to switch to pure strategy s_i.

Proof (define a response function)

- Define a function $f : \boldsymbol{\Delta} \mapsto \boldsymbol{\Delta}$ that satisfies the conditions of Brouwer's fixed point theorem.
- For all $\boldsymbol{x} \in \boldsymbol{\Delta}$, $\boldsymbol{y} = f(\boldsymbol{x})$ where for all $i \in N$ and $s_i \in S_i$,

$$y_i(s_i) := \frac{x_i(s_i) + \mathrm{Gain}_{i;s_i}(\boldsymbol{x})}{1 + \sum_{s_i' \in S_i} \mathrm{Gain}_{i;s_i'}(\boldsymbol{x})}.$$

  - f tries to boost the probability mass that player i places on pure strategies depending on the gains in payoff that the player would get by strategy switching.

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  - $f : \mathbf{\Delta} \mapsto \mathbf{\Delta}$ is continuous (verify this by yourself).
  - $\mathbf{\Delta}$ is a product of simplicies so it is convex (verify this by yourself).
  - $\mathbf{\Delta}$ is closed and bounded, so it is compact.

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⋆ Brouwer's fixed point theorem can ensure the existence of a fixed point of f.

Claim: Any fixed point of f is a Nash equilibrium

- It suffices to prove that a fixed point $\boldsymbol{x} = f(\boldsymbol{x})$ satisfies:
  - $\text{Gain}_{i;s_i}(\boldsymbol{x}) = 0$, for each $i \in N$ and each $s_i \in S_i$.

Claim: Any fixed point of f is a Nash equilibrium

Prove it by contradiction.

- Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p:
  - $\text{Gain}_{p;s_p}(\boldsymbol{x}) > 0$.

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  - $\text{Gain}_{p;s_p}(\boldsymbol{x}) > 0$.
- Note that we must have $x_p(s_p) > 0$, otherwise $\boldsymbol{x}$ cannot be a fixed point of f.
  - From the definition of f; the numerator would be > 0.

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- Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p:
  - $\text{Gain}_{p;s_p}(\boldsymbol{x}) > 0$.
- We argue that there must be some other pure strategy $\hat{s}_p$ such that:
  - $x_p(\hat{s}_p) > 0$ and
  - $u_p(\hat{s}_p; \boldsymbol{x}_{-p}) - u_p(\boldsymbol{x}) < 0$.

  ⋆ Notice that

$$u_p(\boldsymbol{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \boldsymbol{x}_{-p}).$$

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  - $x_p(\hat{s}_p) > 0$ and
  - $u_p(\hat{s}_p; \boldsymbol{x}_{-p}) - u_p(\boldsymbol{x}) < 0 \quad \Rightarrow \quad \text{Gain}_{p,\hat{s}_p}(\boldsymbol{x}) = 0$.
  - ⋆ Notice that
$$u_p(\boldsymbol{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \boldsymbol{x}_{-p}).$$
- We obtain that
$$y_p(\hat{s}_p) := \frac{x_p(\hat{s}_p) + \text{Gain}_{p;\hat{s}_p}(\boldsymbol{x})}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(\boldsymbol{x})} < x_p(\hat{s}_p).$$

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- Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p:
  - $\text{Gain}_{p;s_p}(\boldsymbol{x}) > 0$.
- We argue that there must be some other pure strategy $\hat{s}_p$ such that:
  - $x_p(\hat{s}_p) > 0$ and
  - $u_p(\hat{s}_p; \boldsymbol{x}_{-p}) - u_p(\boldsymbol{x}) < 0 \quad \Rightarrow \quad \text{Gain}_{p,\hat{s}_p}(\boldsymbol{x}) = 0$.
  - ⋆ Notice that
$$u_p(\boldsymbol{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \boldsymbol{x}_{-p}).$$
- We obtain that ($\boldsymbol{x}$ is not a fixed point $\Rightarrow\!\!\!\!\not\Leftarrow$)
$$y_p(\hat{s}_p) := \frac{x_p(\hat{s}_p) + \text{Gain}_{p;\hat{s}_p}(\boldsymbol{x})}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(\boldsymbol{x})} < x_p(\hat{s}_p).$$

Outline

An Extension of Brouwer's work

- Focus: set-valued functions.
  - Refer here for further readings.
  - Why do we consider set-valued functions?

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  - Refer here for further readings.
  - Why do we consider set-valued functions?
    - Best-responses.

Upper Semi-Continuous (having a closed graph)

Upper semi-continuous functions

Let

- $\mathbb{P}(X)$: all nonempty, closed, convex subsets of X.
- S: a nonempty, compact, and convex set.

Then the set-valued function $\Phi : S \mapsto \mathbb{P}(S)$ is upper semi-continuous if

> for arbitrary sequences $(\boldsymbol{x}_n)_{n\in\mathbb{N}}, (\boldsymbol{y}_n)_{n\in\mathbb{N}}$ in S, we have
> - $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}_0$,
> - $\lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}_0$,
> - $\boldsymbol{y}_n \in \Phi(\boldsymbol{x}_n)$ for all $n \in \mathbb{N}$,
>
> imply that $\boldsymbol{y}_0 \in \Phi(\boldsymbol{x}_0)$.

Removable discontinuity, Sequentially compact, Bolzano–Weierstrass theorem.

Fixed Point of Set-Valued Functions

Fixed Point (Set-Valued)

A fixed point of a set-valued function $\Phi : S \mapsto \mathbb{P}(S)$ is a point $\boldsymbol{x}^* \in S$ such that $\boldsymbol{x}^* \in \Phi(\boldsymbol{x}^*)$.

Kakutani's Theorem for Simplices

Kakutani's Theorem for Simplices (1941)

If S is an r-dimensional closed simplex in a Euclidean space and $\Phi : S \mapsto \mathbb{P}(S)$ is upper semi-continuous, then Φ has a fixed point.

Kakutani's Fixed-Point Theorem

Kakutani's Fixed-Point Theorem (1941)

If S is a nonempty, compact, convex set in a Euclidean space and $\Phi : S \mapsto \mathbb{P}(S)$ is upper semi-continuous, then Φ has a fixed point.

Kakutani's Fixed-Point Theorem

Kakutani's Fixed-Point Theorem (1941)

If S is a nonempty, compact, convex set in a Euclidean space and $\Phi : S \mapsto \mathbb{P}(S)$ is upper semi-continuous, then Φ has a fixed point.

- We won't go over its proof.
- Instead, we will delve into how it can be used to prove Nash's Theorem from the perspectives of set-valued functions and best-responses.

Outline

Cartesian product of Sets

Cartesian Product

For a family of sets $\{A_i\}_{i \in N}$, $\prod_{i \in N} A_i = A_1 \times A_2 \times \cdots$ denotes the Cartesian product of A_i for $i \in N$.

Profile

for $x_i \in A_i$, then $(x_i)_{i \in N}$ is called a *(strategy) profile*.

Binary Relation

Binary Relation

- A binary relation on a set A is a subset of $A \times A$ consisting of all pairs of elements.
- For $a, b \in A$, we denote by $R(a, b)$ if a is related to b.

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Properties on Binary Relations

- **Completeness**: For all $a, b \in A$, we have $R(a, b)$, $R(b, a)$, or both.
- **Reflexivity**: For all $a \in A$, we have $R(a, a)$.
- **Transitivity**: For $a, b, c \in A$, if $R(a, b)$ and $R(b, c)$, then we have $R(a, c)$.

Preference Relation

Preference Relation

A preference relation is a complete, reflexive, and transitive binary relation.

- Denote by $a \succsim b$ if a is related to b.
- Denote by $a \succ b$ if $a \succsim b$ but $b \not\succsim a$.
- Denote by $a \sim b$ if $a \succsim b$ and $b \succsim a$.

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- Denote by $a \sim b$ if $a \succsim b$ and $b \succsim a$.

- $a \succsim b$: a is weakly preferred to b.
- $a \sim b$: agent is indifferent between a and b.

Continuity on a Preference relation

Continuous Preference Relation

A preference relation is continuous if:

whenever there exist sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ in A such that

- $\lim_{k\to\infty} a_k = a$,
- $\lim_{k\to\infty} b_k = b$,
- and $a_k \succsim b_k$ for all $k \in \mathbb{N}$

we have $a \succsim b$.

Strategic Games

Strategic Games

A strategic game is a tuple $\langle N, (A_i), (\succsim_i)\rangle$ consisting of

- a finite set of **players** N.
- for each player $i \in N$, a nonempty set of **actions** A_i.
- for each player $i \in N$, a **preference relation** $\succsim_i$ on $A = \prod_{j \in N} A_j$.

- A strategic is finite if A_i is finite for all $i \in N$.

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- A strategic is finite if A_i is finite for all $i \in N$.
- **Note**: $\succsim_i$ is not defined on A_i only, but instead on the set of all $(A_j)_{j \in N}$.

PNE w.r.t. a Preference Relation

Pure Nash Equilibrium (PNE) with $(\succsim_i)$

A (pure) Nash equilibrium (PNE) of a strategic game $\langle N, (A_i), (\succsim_i)\rangle$ is a profile $\boldsymbol{a}^* := (a_i)_{i \in N}$ such that for all $i \in N$, we have

$$(\boldsymbol{a}^*_{-i}, a^*_i) \succsim_i (\boldsymbol{a}^*_{-i}, a'_i) \text{ for all } a'_i \in A.$$

Best-Response Function

Best-Response Functions

The best-response function of player i,

$$BR_i := \prod_{j \in N \setminus \{i\}} A_j \mapsto \mathbb{P}(A_i),$$

is given by

$$BR_i(\boldsymbol{a}_{-i}) = \{a_i \in A_i \mid (\boldsymbol{a}_{-i}, a_i) \succsim_i (\boldsymbol{a}_{-i}, a_i') \text{ for all } a_i' \in A_i\}.$$

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- BR_i is set-valued.

PNE w.r.t. a Preference Relation

- Alternative definition of NE.

Pure Nash Equilibrium (PNE) with $(\succsim_i)$

A Nash equilibrium of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $\boldsymbol{a}^* := (a_i)_{i \in N}$ such that $a_i^* \in BR_i(\boldsymbol{a}_{-i}^*)$ for all $i \in N$.

- Thus, to prove the existence of a NE for a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, it suffices to show that:

PNE w.r.t. a Preference Relation

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A Nash equilibrium of a strategic game $\langle N, (A_i), (\succsim_i)\rangle$ is a profile $\boldsymbol{a}^* := (a_i)_{i\in N}$ such that $a_i^* \in BR_i(\boldsymbol{a}_{-i}^*)$ for all $i \in N$.

- Thus, to prove the existence of a NE for a strategic game $\langle N, (A_i), (\succsim_i)\rangle$, it suffices to show that:
  - There exists a profile $\boldsymbol{a}^* \in A$ such that for all $i \in N$ we have $a_i^* \in BR_i(\boldsymbol{a}_{-i}^*)$.

Outline

General Idea

- Let $BR : A \mapsto \mathbb{P}(A)$ be

$$BR(\boldsymbol{a}) = \prod_{i \in N} BR_i(\boldsymbol{a}_{-i}).$$

- We see for some $\boldsymbol{a}^* \in A$ such that $\boldsymbol{a}^* \in BR(\boldsymbol{a}^*)$.

General Idea

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- We can then use Kakutani's Fixed-Point Theorem to show that $\boldsymbol{a}^*$ exists.

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- We see for some $\boldsymbol{a}^* \in A$ such that $\boldsymbol{a}^* \in BR(\boldsymbol{a}^*)$.
- We can then use Kakutani's Fixed-Point Theorem to show that $\boldsymbol{a}^*$ exists.
- Yet, we need to verify the conditions under which Kakutani's Fixed-Point Theorem holds.

Quasi-Concave

Quasi-Concave of $\succsim_i$

A preference relation $\succsim_i$ over A is quasi-concave on A_i if for all $\boldsymbol{a} \in A$, the set

$$\{a_i' \in A_i \mid (\boldsymbol{a}_{-i}, a_i') \succsim_i (\boldsymbol{a}_{-i}, a_i)\}$$

is convex.

- Then, we can consider the following theorem which guarantees the condition of a PNE.

The Main Theorem I

Main Theorem I

The strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a (pure) Nash equilibrium if

- A_i is a nonempty, compact, and convex subset of a Euclidean space
- $\succsim_i$ is continuous and quasi-concave on A_i for all $i \in N$.

- We will show that A (cf. S) and BR (cf. Φ) satisfy the conditions to apply Kakutani's Fixed-Point Theorem.

Requirements for A & BR

- A_i is nonempty, compact and convex for all $i \in N$, so their Cartesian product (i.e., A) must also be nonempty, compact and convex.

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- Note that in Kakutani's Theorem, $\Phi : S \mapsto \mathbb{P}(S)$, where $\mathbb{P}(S)$ denotes all nonempty, closed, and convex subsets of S.
- We need to show that $BR_i(\boldsymbol{a}_{-i})$ is nonempty, closed, and convex for all $\boldsymbol{a}_{-i} \in \prod_{j \in N \setminus \{i\}} A_j$.
  - Then their Cartesian product $BR(\boldsymbol{a})$ would be nonempty, closed and convex, too.
  - We would then have $BR : A \mapsto \mathbb{P}(A)$.

$BR_i(\boldsymbol{a}_{-i})$ is nonempty

- Assume that we can construct a continuous function (utility function) $u_i : A_i \mapsto \mathbb{R}$ such that for $a_i, a_i' \in A_i$, $(\boldsymbol{a}_{-i}, a_i) \succsim (\boldsymbol{a}_{-i}, a_i')$ if and only if $u_i(a_i) \geq u_i(a_i')$.

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- Since A_i is compact and u_i is continuous, $u_i(A_i)$ is compact as well.
- By the Extreme Value Theorem, there must exist some $a_i^* \in A_i$ such that $u_i(a_i^*) \geq u_i(a_i)$ for all $a_i \in A_i$.

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- By the Extreme Value Theorem, there must exist some $a_i^* \in A_i$ such that $u_i(a_i^*) \geq u_i(a_i)$ for all $a_i \in A_i$.
- By definition of u_i, it follows that $(\boldsymbol{a}_{-i}, a_i^*) \succsim (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$, thus $a_i^* \in BR_i(\boldsymbol{a}_{-i})$.

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- By definition of u_i, it follows that $(\boldsymbol{a}_{-i}, a_i^*) \succsim (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$, thus $a_i^* \in BR_i(\boldsymbol{a}_{-i})$.
- So $BR_i(\boldsymbol{a}_{-i})$ is nonempty.

$BR_i(a_{-i})$ is closed

- Take an arbitrary $p \in \overline{BR_i(\boldsymbol{a}_{-i})}$.
- There must exist some sequence $(p_k)_{k \in \mathbb{N}}$ such that $p_k \in BR_i(\boldsymbol{a}_{-i})$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} p_k = p$.

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- By the definition of BR_i, we know that $(\boldsymbol{a}_{-i}, p_k) \succsim_i (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$.

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- By the definition of BR_i, we know that $(\boldsymbol{a}_{-i}, p_k) \succsim_i (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$.
- For each $a_i \in A_i$, we can construct
  - a sequence $((\boldsymbol{a}_{-i}, p_k))_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(\boldsymbol{a}_{-i}, p_k) = (\boldsymbol{a}_{-i}, p)$.
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- By the definition of BR_i, we know that $(\boldsymbol{a}_{-i}, p_k) \succsim_i (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$.
- For each $a_i \in A_i$, we can construct
  - a sequence $((\boldsymbol{a}_{-i}, p_k))_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(\boldsymbol{a}_{-i}, p_k) = (\boldsymbol{a}_{-i}, p)$.
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- Note that $(\boldsymbol{a}_{-i}, p_k) \succsim_i (\boldsymbol{a}_{-i}, a_i)$ for all $k \in \mathbb{N}$.
  - By the continuity of $\succsim_i$, we have $(\boldsymbol{a}_{-i}, p) \succsim_i (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$.

$BR_i(\boldsymbol{a}_{-i})$ is closed

- Take an arbitrary $p \in \overline{BR_i(\boldsymbol{a}_{-i})}$.
- There must exist some sequence $(p_k)_{k\in\mathbb{N}}$ such that $p_k \in BR_i(\boldsymbol{a}_{-i})$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} p_k = p$.
- By the definition of BR_i, we know that $(\boldsymbol{a}_{-i}, p_k) \succsim_i (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$.
- For each $a_i \in A_i$, we can construct
  - a sequence $((\boldsymbol{a}_{-i}, p_k))_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(\boldsymbol{a}_{-i}, p_k) = (\boldsymbol{a}_{-i}, p)$.
  - a sequence $((\boldsymbol{a}_{-i}, a_i))_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(\boldsymbol{a}_{-i}, a_i) = (\boldsymbol{a}_{-i}, a_i)$.
- Note that $(\boldsymbol{a}_{-i}, p_k) \succsim_i (\boldsymbol{a}_{-i}, a_i)$ for all $k \in \mathbb{N}$.
  - By the continuity of $\succsim_i$, we have $(\boldsymbol{a}_{-i}, p) \succsim_i (\boldsymbol{a}_{-i}, a_i)$ for all $a_i \in A_i$.

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  $\Rightarrow$ $p \in BR_i(\boldsymbol{a}_{-i})$ ($\because BR_i(\boldsymbol{a}_{-i})$ is closed).

$BR_i(\boldsymbol{a}_{-i})$ is convex

- Consider $a_i \in BR_i(\boldsymbol{a}_{-i})$.
- $\succsim_i$ is quasi-concave on $A_i \quad \Rightarrow$

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$$S = \{a_i' \in A_i \mid (\boldsymbol{a}_{-i}, a_i') \succsim_i (\boldsymbol{a}_{-i}, a_i)\} \text{ is convex}$$

- Since a_i is a best response, the responses a_i' weakly preferable to a_i must be also best responses.

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- $\succsim_i$ is quasi-concave on A_i $\Rightarrow$

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- Since a_i is a best response, the responses a_i' weakly preferable to a_i must be also best responses. $\Rightarrow S \subseteq BR_i(\boldsymbol{a}_{-i})$.
- Any other best response $a_i^* \in BR_i(\boldsymbol{a}_{-i})$ must be at least good as a_i $\Rightarrow$ $BR_i(\boldsymbol{a}_{-i}) \subseteq S$.
- Hence, we have $BR_i(\boldsymbol{a}_{-i}) = S$, so $BR_i(\boldsymbol{a}_{-i})$ is convex.

- Next, we will show that BR is upper semi-continuous.

Recall: Upper Semi-Continuous

Upper semi-continuous functions

Let

- $\mathbb{P}(X)$: all nonempty, closed, convex subsets of X.
- S: a nonempty, compact, and convex set.

Then the set-valued function $\Phi : S \mapsto \mathbb{P}(S)$ is upper semi-continuous if

for arbitrary sequences $(\boldsymbol{x}_n)_{n\in\mathbb{N}}, (\boldsymbol{y}_n)_{n\in\mathbb{N}}$ in S, we have

- $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}_0$,
- $\lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{y}_0$,
- $\boldsymbol{y}_n \in \Phi(\boldsymbol{x}_n)$ for all $n \in \mathbb{N}$,

imply that $\boldsymbol{y}_0 \in \Phi(\boldsymbol{x}_0)$.

BR is upper semi-continuous

- Consider two sequences $(\boldsymbol{x}^k), (\boldsymbol{y}^k)$ in A such that

  $\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^0$,
  $\lim_{k\to\infty} \boldsymbol{y}^k = \boldsymbol{y}^0$.
  $\boldsymbol{y}^k \in BR_i(\boldsymbol{x}^k)$ for all $k \in \mathbb{N}$.

- Then we have $y_i^k \in BR_i(\boldsymbol{x}_{-i}^k)$ for all $i \in N, k \in \mathbb{N}$.

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  $\lim_{k\to\infty} \boldsymbol{y}^k = \boldsymbol{y}^0$.
  $\boldsymbol{y}^k \in BR_i(\boldsymbol{x}^k)$ for all $k \in \mathbb{N}$.
- Then we have $y_i^k \in BR_i(\boldsymbol{x}_{-i}^k)$ for all $i \in N, k \in \mathbb{N}$.
- For an arbitrary $i \in N$, we have $(\boldsymbol{x}_{-i}^k, y_i^k) \succsim_i (\boldsymbol{x}_{-i}^k, a_i)$ for all $a_i \in A_i$ and $k \in \mathbb{N}$ ($\because$ best response).

BR is upper semi-continuous (contd.)

- For each $a_i \in A_i$, we can construct:
  - a sequence $((\boldsymbol{x}_{-i}^k, y_i^k))_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(\boldsymbol{x}_{-i}^k, y_i^k) = (\boldsymbol{x}_{-i}^0, y_i^0)$.
  - a sequence $((\boldsymbol{x}_{-i}^k, a_i))_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(\boldsymbol{x}_{-i}^k, a_i) = (\boldsymbol{x}_{-i}^0, a_i)$.
- Note that we have $(\boldsymbol{x}_{-i}^k, y_i^k) \succsim_i (\boldsymbol{x}_{-i}^k, a_i)$ for all $k \in \mathbb{N}$.
  - By continuity of $\succsim_i$, we have $(\boldsymbol{x}_{-i}^0, y_i^0) \succsim_i (\boldsymbol{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.

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  - By continuity of $\succsim_i$, we have $(\boldsymbol{x}_{-i}^0, y_i^0) \succsim_i (\boldsymbol{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.
- Thus, we have $y_i^0 \in BR_i(\boldsymbol{x}_{-i}^0)$ for all $i \in N$.
  - $\boldsymbol{y}^0 \in BR_i(\boldsymbol{x}^0)$.

BR is upper semi-continuous (contd.)

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  - By continuity of $\succsim_i$, we have $(\boldsymbol{x}_{-i}^0, y_i^0) \succsim_i (\boldsymbol{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.
- Thus, we have $y_i^0 \in BR_i(\boldsymbol{x}_{-i}^0)$ for all $i \in N$.
  - $\boldsymbol{y}^0 \in BR_i(\boldsymbol{x}^0)$.
- Therefore, BR is upper semi-continuous.

BR is upper semi-continuous (contd.)

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  - By continuity of $\succsim_i$, we have $(\boldsymbol{x}_{-i}^0, y_i^0) \succsim_i (\boldsymbol{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.
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  - $\boldsymbol{y}^0 \in BR_i(\boldsymbol{x}^0)$.
- Therefore, BR is upper semi-continuous.

  By Kakutani's Fixed-Point Theorem, there exists some $\boldsymbol{a}^* \in A$ such that $\boldsymbol{a}^* \in BR(\boldsymbol{a}^*)$

BR is upper semi-continuous (contd.)

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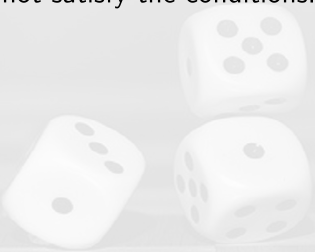
- Thus, we have $y_i^0 \in BR_i(\boldsymbol{x}_{-i}^0)$ for all $i \in N$.
  - $\boldsymbol{y}^0 \in BR_i(\boldsymbol{x}^0)$.
- Therefore, BR is upper semi-continuous.

  By Kakutani's Fixed-Point Theorem, there exists some $\boldsymbol{a}^* \in A$ such that $\boldsymbol{a}^* \in BR(\boldsymbol{a}^*) \ \Rightarrow \boldsymbol{a}^*$ is a PNE of the strategic game.

Outline

Limitations of the Previous PNE Result

- Any finite game cannot satisfy the conditions.



Limitations of the Previous PNE Result

- Any finite game cannot satisfy the conditions.
  - Each A_i cannot be convex if it is finite and nonempty.

⋆ Next, we consider extending finite games into non-deterministic (randomized) strategies.

Assumptions

- For a strategic game $\langle N, (A_i), (\succsim_i)\rangle$, we assume that we can construct a utility function $u_i : A \mapsto \mathbb{R}$, where $A = \prod_{i \in N} A_i$.
- Each player's *expected utility* is coupled with the set of probability distributions over A.
- $\Delta(X)$: the set of probability distributions over X.
- If X is finite and $\delta \in \Delta(X)$, then
  - $\delta(x)$: the probability that δ assigns to $x \in X$.
  - The support of δ: $\chi(\delta) = \{x \in X \mid \delta(x) > 0\}$.

Mixed Strategy

Mixed Strategy

Given a strategic game $\langle N, (A_i), (u_i) \rangle$, we call

- $\alpha_i \in \Delta(A_i)$ a mixed strategy.
- $a_i \in A_i$ a pure strategy.

A profile of mixed strategies $\boldsymbol{\alpha} = (\alpha_j)_{j \in N}$ induces a probability distribution over A.

- The probability of $\boldsymbol{a} = (a_j)_{j \in N}$ under $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}(\boldsymbol{a}) = \prod_{j \in N} \alpha_j(a_j). \quad \text{(a normal product)}$$

(A_i is finite $\forall i \in N$ and each player's strategy is resolved independently.)

Mixed Extension of $\langle N, (A_i), (u_i)\rangle$

Mixed Extension of the Strategic Games

$\langle N, (\Delta(A_i)), (U_i)\rangle$:

- $U_i : \prod_{i\in N} \Delta(A_i) \mapsto \mathbb{R}$; expected utility over A induced by $\boldsymbol{\alpha} \in \prod_{i\in N} \Delta(A_i)$.
- If A_j is finite for all $j \in N$, then

$$
\begin{aligned}
U_i(\boldsymbol{\alpha}) &= \sum_{\boldsymbol{a}\in A} (\boldsymbol{\alpha}(\boldsymbol{a}) \cdot u_i(\boldsymbol{a})) \\
&= \sum_{\boldsymbol{a}\in A} \left(\left(\prod_{j\in N} \alpha_j(a_j) \right) \cdot u_i(\boldsymbol{a}) \right).
\end{aligned}
$$

Main Theorem II

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Every finite strategies game has a mixed strategy Nash equilibrium.

- Consider an arbitrary finite strategic game $\langle N, (A_i), (u_i) \rangle$, let $m_i := |A_i|$ for all $i \in N$.
- Represent each $\Delta(A_i)$ as a collection of vectors $\boldsymbol{p}^i = (p_1, p_2, \ldots, p_{m_i})$.
  - $p_k \geq 0$ for all $k \in [m_i]$ and $\sum_{k=1}^{m_i} p_k = 1$.
  - $\Delta(A_i)$ is a standard $m_i - 1$ simplex for all $i \in N$.

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  - $p_k \geq 0$ for all $k \in [m_i]$ and $\sum_{k=1}^{m_i} p_k = 1$.
  - $\Delta(A_i)$ is a standard $m_i - 1$ simplex for all $i \in N$.
  - ⋆ $\Delta(A_i)$: nonempty, compact, and convex for each $i \in N$.
- U_i: continuous ($\because$ multilinear).
- Next, we show that U_i is quasi-concave in $\Delta(A_i)$.

Proof of Main Theorem II (contd.)

- Consider $\boldsymbol{\alpha} \in \prod_{i \in N} \Delta(A_i)$.
- **Goal:** Show that $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\boldsymbol{\alpha}_{-i}, \alpha'_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)\}$ is convex.

Proof of Main Theorem II (contd.)

- Consider $\boldsymbol{\alpha} \in \prod_{i \in N} \Delta(A_i)$.
- **Goal:** Show that $S = \{\alpha_i' \in \Delta(A_i) \mid U_i(\boldsymbol{\alpha}_{-i}, \alpha_i') \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)\}$ is convex.
- Take $\beta_i, \gamma_i \in S$, $\lambda \in [0, 1]$.
- By definition of S, we have
  - $U_i(\boldsymbol{\alpha}_{-i}, \beta_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$, and
  - $U_i(\boldsymbol{\alpha}_{-i}, \gamma_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$.

Proof of Main Theorem II (contd.)

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- **Goal:** Show that $S = \{\alpha_i' \in \Delta(A_i) \mid U_i(\boldsymbol{\alpha}_{-i}, \alpha_i') \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)\}$ is convex.
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- By definition of S, we have
  - $U_i(\boldsymbol{\alpha}_{-i}, \beta_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$, and
  - $U_i(\boldsymbol{\alpha}_{-i}, \gamma_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$.
- $\lambda U_i(\boldsymbol{\alpha}_{-i}, \beta_i) + (1-\lambda) U_i(\boldsymbol{\alpha}_{-i}, \gamma_i) \geq \lambda U_i(\boldsymbol{\alpha}_{-i}, \alpha_i) + (1-\lambda) U_i(\boldsymbol{\alpha}_{-i}, \alpha_i) = U_i(\boldsymbol{\alpha}_{-i}, \alpha_i)$.

Proof of Main Theorem II (contd.)

- By the multilinearity of U_i, we have

$$\lambda U_i(\boldsymbol{\alpha}_{-i}, \beta_i) + (1-\lambda)U_i(\boldsymbol{\alpha}_{-i}, \gamma_i) = U_i(\boldsymbol{\alpha}_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i).$$

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- So,

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$$\lambda\beta_i + (1-\lambda)\gamma_i \in S$$

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- So,

$$U_i(\boldsymbol{\alpha}_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i) \geq U_i(\boldsymbol{\alpha}_{-i}, \alpha_i).$$

$$\lambda\beta_i + (1-\lambda)\gamma_i \in S \Rightarrow \; U_i \text{ is convex.}$$

- Thus, U_i is quasi-concave in $\Delta(A_i)$.

  We are done.